MAT4701 - STOCHASTIC ANALYSIS AND APPLICATIONS

ASSIGNMENT

Exercise 1: Itô Isometry
In the construction of the Itô non-anticipating stochastic integral a crucial tool is the so-called Itô isometry. In this exercise, prove the Itô isometry for simple integrands

$$E\Big[(I_t(\phi))^2\Big] = E\Big[\int_0^T \int_E (\phi(t,x))^2 \rho(dt,dx)\Big]$$

in the case of integration with respect to

- Brownian motion, i.e. $M(dt,dx) = dB(t)\delta_{\{0\}}(dx)$
- a martingale measure $M(dt,dx)$ (see the notation and the concept of *real-valued independently scattered martingale valued measure* in [Applebaum, Chapter 4]).

In both cases provide a preliminary description of the integrator and the simple integrands. Give details in your computations.

Exercise 2: Application of conditional expectation
In both the cases of integration with respect to Brownian motion and with respect to a general martingale measure (as above), prove the following properties of the Itô non-anticipating stochastic integral $I_T(\phi) = \int_0^T \int_{\mathbb{R}} \phi(t,x) M(dt,dx)$:

- $E[I_T(\phi)] = 0$
- $E\big[I_T(\phi) - I_t(\phi) | \mathcal{F}_t\big] = 0$

- $$E\Big[(I_T(\phi) - I_t(\phi))^2|\mathcal{F}_t\Big] = E\Big[\int_t^T \int_E (\phi(t,x))^2 \rho(dt,dx)|\mathcal{F}_t\Big]$$

Exercise 3: Around Brownian motion

Let W be a Brownian motion. Use the Lévy martingale characteriation of a Brownian motion (see [Applebaum, Theorem 2.2.7]) to show that $V(t) := \frac{1}{c}W(c^2t)$, $t \geq 0$ is a Brownian motion.

Prove the same from the definition of Brownian motion.