Lecture notes 21: Nucleosynthesis. Measuring Cosmological Parameters

In the last lecture, using the ?uid equation, we derived a relation between the energy density of a material obeying an equation of state $P = w$ and the expansion factor a

$$ _w = \ _{w,0} a^{-3(1+w)}. \tag{1}$$

This implies that non-relativistic matter ($w = 0$) decreases in energy density as

$$ _m(a) = \ _{m,0}/a^3, \tag{2}$$

while radiation ($w = 1/3$) follows

$$ _r(a) = \ _{r,0}/a^4. \tag{3}$$

Why a^{-4} for radiation? The number density of photons goes down as a^{-3} as does regular matter, but in addition each individual photon loses energy as it?s wavelength is ?stretched? by the expansion of the Univserse.

The current benchmark model of the Universe has a total radiation content of $\Omega_{r,0} = 8.4 ? \ 10^{-5}$, including both neutrinos and photons, and a total matter content of $\Omega_{m,0} = 0.3$, including both baryonic matter and dark matter. The ratio between these today is then $\Omega_{r,0}/\Omega_{m,0} = 0.00028$. However, as we follow the Universes evolution backwards in time towards steadily smaller a we ?nd that this ratio must grow as $1/a(t)$. Thus, at time t such that $1 + z = 1/a(t) \approx 3571$ we have matter ? radiation equality, and at times before this the Universe was dominated by radiation.

In a radiation dominated Universe we have proved that $a(t) \quad t^{1/2}$ and since the temperature of blackbody photons in the univserse decrease as $T \quad 1/a$, we have

$$T(t) \approx 10^{10}\ \mathrm{K}\left(\frac{t}{1\ \mathrm{s}}\right)^{-1/2} \tag{4}$$

which equivalent to

$$kT(t) \approx 1\ \mathrm{MeV}\left(\frac{t}{1\ \mathrm{s}}\right)^{-1/2}. \tag{5}$$

The ?rst three minutes

In the classic book *The ?rst three minutes* by Steven Weinberg the story of the creation of the material elements is described. We will pick up the story at $t < 10^{-4}$ s at which time the energy of the photons is on the order 150 Mev. At this energy quarks are not con?ned within baryons or other particles but instead formed a mass particles, along with the leptons and photons, sometimes called a *quark?gluon soup*. In this state photon interactions were energetic enough to produce quark ? antiquark interactions

$$\gamma + \gamma \leftrightarrow q + \bar{q} \tag{6}$$

Where the q and $\bar{q}$ could represent for expample up and anti-up or down and anti-down quarks. As the temperature as the soup falls the energy of photons is reduced and no longer su cient to replace the quarks removed by particle ? antiparticle annihilations. However, for some reason there is a slight asymmetry in the laws of physics ? as pointed out by Andrei Sakharov already in 1967 ? between matter and antimatter and some quarks are left over after the bulk of the quark ? antiquark pairs have annihilated. The ratio of photons to baryons found in the Universe today give a clue to size of this asymmetry. We will later show how we can use the theory of Big Bang nucleosynthesis to measure this asymmetry.

At $t \approx 0.1$ s the quarks are bound in baryons but there is still su cient energy in the photon ?uid to allow the reaction

$$\gamma + \gamma \leftrightarrow {}^{0}_{-1}\mathrm{e} + {}^{1}_{1}\mathrm{p} \tag{7}$$

A free neutron is unstable, decaying by the process

$$ {}^{1}_{0}\mathrm{n} \rightarrow {}^{1}_{1}\mathrm{p} + {}^{0}_{-1}\mathrm{e} + {}^{0}_{0}\bar{\nu} \tag{8}$$

on a decay time of $\tau_{\mathrm{n}} = 890$ s. However at these temperatures the relative populations of neutrons and protons is maintained at thermodynamic equilibrium by the two processes

$$ {}^{1}_{0}\mathrm{n} + {}^{0}_{0}\nu \leftrightarrow {}^{1}_{1}\mathrm{p} + {}^{0}_{-1}\mathrm{e} \tag{9}$$

and

$$ {}^{1}_{0}\mathrm{n} + {}^{0}_{1}\bar{\mathrm{e}} \leftrightarrow {}^{1}_{1}\mathrm{p} + {}^{0}_{0}\bar{\nu}. \tag{10}$$

As long as these processes are in thermodynamic equilibrium the neutron to proton ratio is given by Bolztmann statistics as

$$\frac{n_n}{n_p} = (\frac{m_n}{m_p})^{3/2} \exp \left(? \frac{(m_n ? m_p)c^2}{kT} \right) \tag{11}$$

The di erence in the rest energies between protons and neutrons is $(m_n ? m_p)c^2 = 1.29$ MeV and as the temperature falls below some $1.5 ? 10^{10}$ K and the age of the Universe approaches 1 s the number of neutrons falls rapidly as long as thermodynamic equilibrium perists. However, the neutrons and protons do not remain in this state for long since the reactions that mediate between protons and neutrons depend on neutrinos and antineutrinos and thus the weak force. The cross section for these reactions is $\sigma_w \quad T^2$ and at the temperatures we are considering here the cross section is also small

$$\sigma_w \quad 10^{-47}\ \mathrm{m}^2 (\frac{kT}{1\ \mathrm{Mev}})^2 \tag{12}$$

(compare to the Thomson cross section for electrons via the electronmagnetic force of $\sigma_e = 6.65 ? 10^{29}\ \mathrm{m}^2$). This means that the cross section $\sigma_w \quad t^{-1}$; while at the same time the number density of neutrinos falls as $n_\nu \quad a(t)^{-3} \quad t^{-3/2}$. This means that the interaction rate Γ falls as

$$\Gamma = n_\nu c \sigma_w \quad t^{-5/2} \tag{13}$$

At the same time the Hubble parameter $H \quad t^{-1}$. When $\Gamma \approx H$ neutrinos decouple from neutrons and protons since at that time the mean free path for a neutrino $l = 1/n_\nu \sigma_w$ becomes longer than the Universe?s horizon c/H; a neutrino can cross the visible universe without colliding. It turns out this occurs at the temperature $9 ? 10^9$ K when the age of the Universe is roughly 1 s. At that temperature we ?nd $n_n/n_p \approx 0.2$, a ratio which is ?frozen? at times short compared to the neutron decay time τ_n.

There were therefore few neutrons ? one for every ?ve protons ? available for nucleosynthesis during the few minutes the Universe was hot and dense enough to support fusion. The processes

$$ {}^1_1\mathrm{p} + {}^1_1\mathrm{p} \leftrightarrow {}^2_1\mathrm{H} + {}^0_1\mathrm{e} + {}^0_0\nu \tag{14} $$

and

$$ {}^1_0\mathrm{n} + {}^1_0\mathrm{n} \leftrightarrow {}^2_1\mathrm{H} + {}^0_{-1}\mathrm{e} + {}^0_0\bar{\nu} \tag{15} $$

are too slow to be e ective on these timescales. (Though the ?rst one *is* important and e ective in the Sun; the Sun has plenty of time to e ect fusion, also by the weak interaction, as the Solare core is an environment with stable density and temperature over billions of years.) Thus fusion proceeds via deuterium via the strong interaction process

$$ {}^1_1\mathrm{p} + {}^1_0\mathrm{n} \leftrightarrow {}^2_1\mathrm{H} + {}^0_0\gamma. \tag{16} $$

But with two neutrons for every 10 protons this means that we can produce at most a Helium fraction (one ${}^4_2$He atom requires two neutrons) $Y_{\mathrm{max}} = 4/12 = 1/3 = 0.33$; the theory presented above would be in great trouble should a greater primordial fraction be observed. Luckily the observed ratio is $Y \approx 0.24$, so we seem to be on the right track.

While neutrinos are decoupled, photons and matter are still well coupled at $t \approx 2$ s and this implies that the number density of particles following reaction 16 follow Saha-Boltzmann statistics

$$ \frac{n_D}{n_p n_n} = \frac{g_D}{g_p g_n}\left(\frac{m_D}{m_p m_n}\right)^{3/2}\left(\frac{kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left[\frac{(m_p + m_n ? m_D)c^2}{kT}\right], \tag{17} $$

where the g?s are statistical weights of the various constituents (for deuterium $g_D = 3$, protons and neutrons have $g = 2$). Computing the di erence between the masses gives us the binding energy $B_D = 2.2$ MeV and we may rewrite this equation as

$$ \frac{n_D}{n_n} = 6 n_p \left(\frac{m_n kT}{\pi\hbar^2}\right)^{-3/2} \exp\left[B_D/kT\right], \tag{18} $$

We will now proceed by introducing the ratio between the number of baryons n_b and photons in the Universe $\eta \equiv n_b/n_\gamma$. The number of protons is of the order $0.75 n_b$ in today?s universe, while we have just computed that 5 of 6 baryons, *i.e.* $0.83 n_b$, were protons in the early universe. Let us pick an approximate relation $n_p \approx 0.8 n_b$ and recall the number density of photons in equilibrium at temperature T is $n_\gamma = 0.243(kT/\hbar c)^3$ to rewrite the deuterium ot neutron density as

$$\frac{n_D}{n_p} \approx 6.5\eta(\frac{kT}{m_n c^2})^{3/2} \exp(B_D/kT) \tag{19}$$

At high temperatures free protons and neutrons are preferred, at low temperatures the bound state deuterium is preferred.

We ?nd from the above that half of all neutrons are incorperated into deuterium nuclei at $T \approx 7.6? 10^8$ K, which corresponds to a time $t_{nuc} \approx 200$ s. This time is not insigni?cant compared to τ_n and we must correct for the number of neutrons that have decayed into protons during t_{nuc}:

$$\frac{n_n}{n_p} \approx \frac{\exp(? 200 \text{ s}/890 \text{ s})}{5 + [1 ? \exp(? 200 \text{ s}/890 \text{ s})]} \approx 0.15 \tag{20}$$

This reduces the maximum helium content to $Y_{max} = 0.27$.

Beyond deuterium A number of reactions convert deuterium into varius other atoms: ?rst via the processes

$$\begin{aligned}
{}^2_1\text{H} + {}^1_1\text{p} &\leftrightarrow {}^3_2\text{He} + {}^0_0\gamma \\
{}^2_1\text{H} + {}^1_0\text{n} &\leftrightarrow {}^3_1\text{H} + {}^0_0\gamma \\
{}^2_1\text{H} + {}^2_1\text{H} &\leftrightarrow {}^4_2\text{He} + {}^0_0\gamma \\
{}^2_1\text{H} + {}^2_1\text{H} &\leftrightarrow {}^3_1\text{H} + {}^1_1\text{p} \\
{}^2_1\text{H} + {}^2_1\text{H} &\leftrightarrow {}^3_2\text{He} + {}^1_0\text{n}
\end{aligned}$$

Note that though tritium is unstable with a half life of 18 yr this is a very long time-scale compared to that of Big Bang nucleosynthesis, and tritium can be considered a stable particle for our purposes. The direct production of ${}^4_2$He is not very likely, but helium is produced in copius amounts soon after the incorporation of deuterium into other nuclei;

$$\begin{aligned}
{}^3_1\text{H} + {}^1_1\text{p} &\leftrightarrow {}^4_2\text{He} + {}^0_0\gamma \\
{}^3_2\text{He} + {}^1_0\text{n} &\leftrightarrow {}^4_2\text{He} + {}^0_0\gamma \\
{}^3_1\text{H} + {}^2_1\text{H} &\leftrightarrow {}^4_2\text{He} + {}^1_0\text{n} \\
{}^3_2\text{He} + {}^2_1\text{H} &\leftrightarrow {}^4_2\text{He} + {}^1_1\text{p}
\end{aligned}$$

${}^4_2$He represents a roadblock as there are no stable nuclei with atomic number equal to 5. However, small amounts of ${}^6_3$Li, and ${}^7_3$Li are produced through the processes

$$\begin{aligned}
{}^4_2\text{He} + {}^2_1\text{H} &\leftrightarrow {}^6_3\text{Li} + {}^0_0\gamma \\
{}^4_2\text{He} + {}^3_1\text{H} &\leftrightarrow {}^7_3\text{Li} + {}^0_0\gamma \\
{}^4_2\text{He} + {}^3_2\text{He} &\leftrightarrow {}^7_4\text{Be} + {}^0_0\gamma
\end{aligned}$$

the beryllium produced in the last process eventually decays to lithium via

$${}^7_4\text{Be} + {}^{\ 0}_{-1}\text{e} \rightarrow {}^7_3\text{Li} + {}^0_0\nu \tag{21}$$

. Synthesis to elements with atomic weight > 7 is hindered by there being no stable atomic nuclei with $A = 8$, as we have earlier noted in discussing the triple-α process where helium is fused to carbon in stars the lifetime of the beryllium produced in

$$ {}^{4}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{8}_{4}\mathrm{Be} + {}^{0}_{0}\gamma \qquad (22)$$

is very short lived with a decay time of $\tau = 3 \times 10^{-16}$ s.

Fusion of elements up to helium proceeds very rapidly in the Big Bang, but is hindered by atomic physics from continuing on, and the universe must use the stable conditions found inside stars in order to produce the heavier elements we find ourselves made up of.

A very important note is the following: we found the amount of deuterium formed in the Big Bang to be a function of the baryon to photon ration of the Universe. Thus, measurments of the primordial deuterium abundance can tell us what the baryon density of the universe is. Measurments show that the primordial deuterium to hydrogen ratio is $n_D/n_H = 3.0 \times 10^{-5}$. This number converts to a baryon to photon ratio $\eta = 5.5 \times 10^{-10}$, giving an expected baryon density today of $n_{b,0} = 0.23$ particles/m^3 corresponding to $\Omega_{b,0} = 0.04$.

Measuring $a(t)$

Being able to measure $a(t)$ is a key in determining which components the Universe consists of as should be obvious from the Friedmann equation. Let us now consider how this can be done: A Taylor expansion of $a(t)$ around $t = t_0$ is given by

$$a(t) = a(t_0) + \frac{da}{dt}\Big|_{t=t_0}(t - t_0) + \frac{1}{2}\frac{d^2a}{dt^2}\Big|_{t=t_0}(t - t_0)^2 + \cdots$$

Dividing by $a(t_0)$, remembering that $a(t_0) = 1$ and that $H_0 = \frac{\dot{a}}{a}\big|_{t=t_0}$ gives

$$a(t) \approx 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2 (t - t_0)^2, \qquad (23)$$

where we have defined the **deceleration parameter**

$$q_0 \equiv -\left(\frac{\ddot{a}a}{\dot{a}^2}\right)_{t=t_0} = -\left(\frac{\ddot{a}}{aH^2}\right)_{t=t_0}. \qquad (24)$$

Note the sign of the parameter.

The measurement of the redshift z of a galaxy or other object is relatively simple, the measurement of its distance is difficult. In fact we must be quite careful even in *defining* the distance; perhaps it is best to use the proper distance d_p as a measure of the distance between two points in an expanding universe. Starting from the relation

$$d_p(t_0) = \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

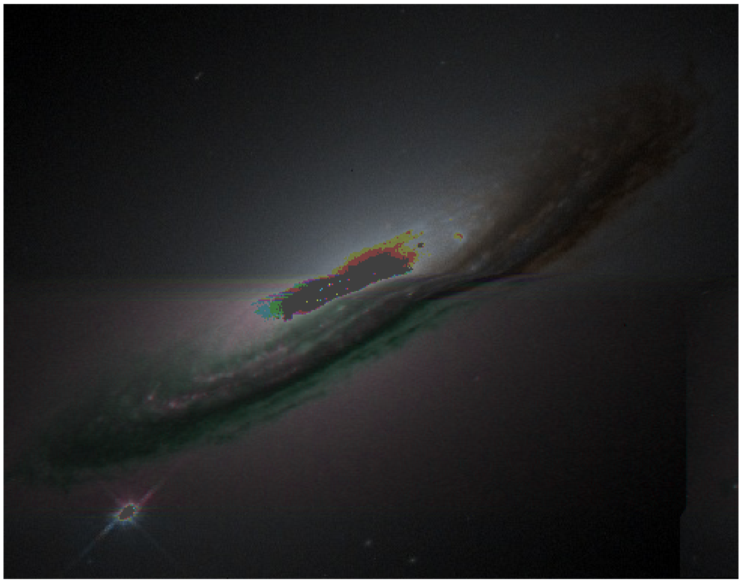

Figure 1: A supernova of type Ia in galaxy NGC 4526.

and equation 23 it is relatively straightforward to arrive at an approximate relation between the proper distance and the redshift of a galaxy

$$d_p(t_0) \approx \frac{c}{H_0}\left[z ? \left(\frac{1+q_0}{2}\right) z^2\right] + \frac{cH_0}{2}\frac{z^2}{H_0^2} = \frac{c}{H_0} z \left[1 ? \frac{1+q_0}{2} z\right] \qquad (25)$$

I.e this is Hubble?s law corrected for the eventual acceleration or deceleration of the Univserse.

Now we must relate the proper distance to actual measurments made in an expanding universe that may or may not be curved. There are in principle two methods of measuring distance; via the luminosity of a ?standard candle?, or via the extent of a ?standard ruler?. To make a long story rather short we can say that the net e ect of universal expansion in a nearly ?at universe is to make the luminosity distance

$$d_L = S_\kappa(r)(1+z) \approx r(1+z) = d_p(t_0)(1+z),$$

or using equation 25 we ?nd

$$d_L \approx \frac{c}{H_0} z \left[1 + \frac{1 ? q_0}{2} z\right].$$

On the other hand the angular-diameter distance is given by

$$d_A = \frac{S_\kappa(r)}{1+z} = \frac{d_L}{(1+z)^2}$$

which for small z can be written

$$d_A \approx \frac{c}{H_0} z \left(1 ? \frac{3 + q_0}{2} z \right).$$

Note that in the limit $z \to \infty$ we have $d_p(t_0) \approx d_{hor}(t_0)$ which gives

$$d_L(z \to \infty) \approx z d_{hor}(t_0) \quad \text{and} \quad d_A(z \to \infty) \approx \frac{d_{hor}(t_0)}{z}$$

The accelerating universe. Traditionally one has used Cepheids, with their well known period-luminosity relation, as standard candles. These stars are fairly bright with luminosities in the range $L = 400 \to 40\,000$ L_S, and can be see over fairly extended distances. One problem with this program is that the parallax distance to these stars is not well known; only two are within measurment range of the Hipparchos satellite, Polaris at $d_\pi = 130 ? 10$ pc and δ Cephei at $d_\pi 300 ? 50$ pc. The modest accuracy of these measurements has been a long standing problem. In any case Cepheid distances may be measured out to luminosity distances of d_L 20 Mpc, for example the Virgo cluster is found to have a distance of some 15 Mpc.

There is, however, another problem associated with this measurement: 20 Mpc is too small a distance for the universe to be homegenous and isotropic. The Local Group is actually falling towards the Virgo cluster at a peculiar speed of some 250 km/s.

During the last decade supernova of type sc Ia have been used as standard candles. These stars have luminosities on the order $L = 4 ? 10^9$ L_S, as bright as their host galaxies, and are observable to distances z 1. The results of these studies is that the universal expansion seems to be accelerating consitent with a cosmological constant of $\Omega_\Lambda = 0.7$.

Dark and light matter

Measurments of the velocity dispersion of galaxy clusters as well as measurements of the gravitational lensing caused by lensing seem to indicate that the total density of matter ? light and dark ? in the universe is of order $\Omega = 0$ .3.

Recombination, Decoupling and the CMB

Let us now return to the early universe, in the period following big bang nucleosynthesis. Photons and luminous matter are coupled through collisions,

$$^0_0\gamma + {}^{0}_{-1}\mathrm{e} \leftrightarrow {}^0_0\gamma + {}^{0}_{-1}\mathrm{e}$$

mainly by Thomson scatterings, so that the temperature is identical in each component. As long as collisions are frequent enough this will be the case. However, when the collisional rate Γ falls below the Hubble parameter H we expect decoupling. The mean free path between collisions is $l = 1/n_e\sigma_e$, where

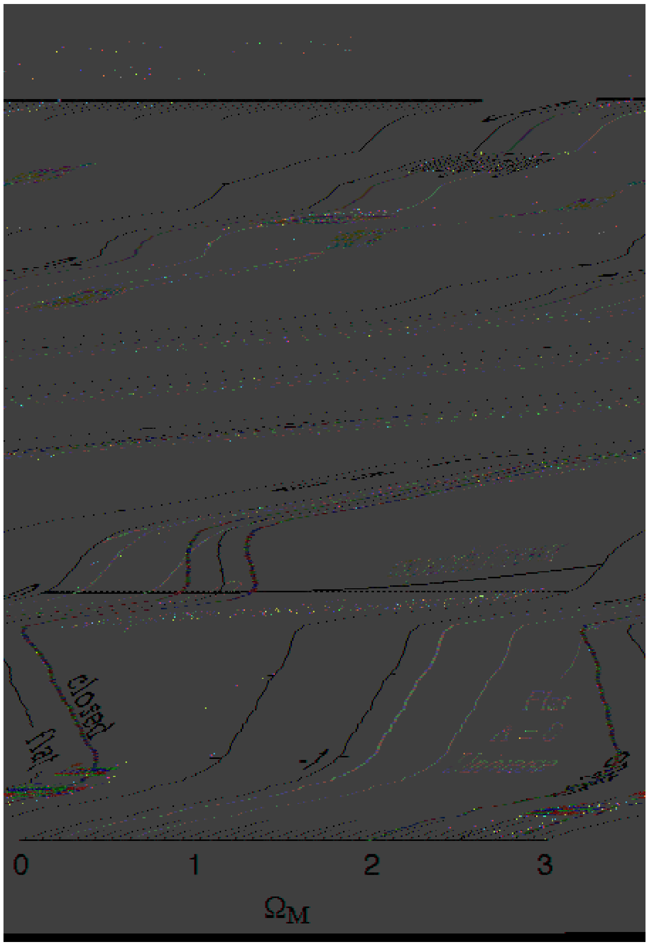

Figure 2: The values of Ω_m and Ω_Λ consitent with cosmological observations.

$\sigma_e = 6.65 ? \ 10^{-29}$ m^2. Thus, photons which travel at speed c will collide at a rate

$$\Gamma = c/l = n_e \sigma_e c$$

In a fully ionized plasma we have (roughly) $n_e = n_p = n_b$, where n_b is the baryon density. Currently we have $n_{b,0} = 0.23$ kg/m^3, and the baryon density scales with the expansion factor as $n_{b,0}/a^3$. *If* hydrogen remains ionized the condition $\Gamma = H$ is not met until relatively recently. However, the averga photon temperature would in that case not be large enough to keep hydrogen ionized; we must take into account the dissapearance of electrons as the average photon temperature falls.

Hydrogen ionization/recombination proceeds as

$$^1_1\mathrm{H} + ^0_0\gamma \leftrightarrow ^1_1\mathrm{p} + ^{\ 0}_{-1}\mathrm{e}$$

In thermodynamic equilbrium the ratio of neutral hydrogen to protons and electrons is described by the Saha-Boltzmann equation

$$\frac{n_H}{n_p n_e} = \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2} \exp[[I_H/kT]$$

Solving this equation for the ionization degree

$$X \equiv \frac{n_p}{n_H + n_p} = \frac{1}{2}$$

gives a recombination temperature of $T_{rec} = 3740$ K, equivalent to $z = 1370$ or a time $t = 240\,000$ yr after the big bang. With the removal of electrons and thus electron scattering of photons decoupling happens rapidly afterwards as may be found by considering

$$\Gamma(z) = n_e(z)\sigma_e c = X(z)(1 + z^3)n_{b,0}\sigma_e c,$$

and the evolution of the Hubble parameter which in this now matter dominated universe is

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} = \Omega_{m,0}(1 + z)^3$$

. or

$$H(z) = H_0\Omega_{m,0}(1 + z)^{3/2}$$

These relations ?nally give a decoupling of radiation and matter at $z_{dec} = 1130$, corresponding to $T_{dec} = 3000$ K, or time $t_{dec} = 350\,000$ yr. This time closely corresponds to the time of the last scattering; *i.e* the age of the universe we see re?ected in the cosmic microwave background (CMB).

Temperature ?uctuations in the CMB

Measurements of the size of temperature ?uctuations in the CMB can give us an idea of the geometry of space-time: if the scale of the ?uctuations appear larger than they should in a ?at universe the universe is open, if they appear smaller the universe is closed. One structure of known size is the Hubble distance at the time or redshift of the last scattering $c/H(z_{ls} \approx 0.2$ Mpc, using $z_{ls} = 1100$. The angular size of this structure should be of order

$$\theta_H = \frac{c/H(z_{ls}}{d_A} \approx \frac{0.2\ \mathrm{Mpc}}{13\ \mathrm{Mpc}} \approx 0.015\ rad \approx 1^o$$

where we have used the relation $d_A \approx d_{hor}(t_0)/z_{ls}$.

There are two e ects that give temperature variations in the CMB:

1. The *Sachs-Wolfe e ect* caused by photons being redshifted as they climb out of the gravitational potential wells caused by irregularities in the dark matter or conversely the blueshift caused by photons gaining energy as they fall down a potential ?hill?.

2. On smaller scales the photon-baryon ?uid moves inward and outward around under the gravitational in?uence of dark matter causing acoustic oscillations and associated gains and losses of energy caused by compression and rarefacation.

As shown in ?gure 2 measurements of the size of (the largest) irregularities of the CMB indicate that the universe is close to ?at, *i.e.* that $1 ? \Omega \approx 1$.